

Quantum- and quasi-Plücker coordinates

Aaron Lauve¹

*Department of Mathematics
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway NJ, 08854-8019, USA*

Abstract

We demonstrate a passage from the “quasi-Plücker coordinates” of Gelfand and Retakh, to the quantum Plücker coordinates built from q -generic matrices. In the process, we rediscover the defining relations of the quantum Grassmannian of Taft and Towber and provide that algebra with more concrete geometric origins.

Key words: quantum group, quantum minor, Grassmannian, quasideterminant
1991 MSC: 20G42, 16S38, 15A15

Introduction

Since the problem of constructing quantum flag and Grassmann spaces was first posed in Manin’s Montréal lectures [17], numerous approaches to the problem have appeared. In this paper, we focus on the efforts of Lakshmibai-Reshetikhin [14] and Taft-Towber [22] to build the quantized homogeneous coordinate ring $\mathcal{G}_q(d, n)$ of the Grassmannian of d -dimensional subspaces in K^n . The difficulty lies in attaching good geometric data to any algebraic structure proposed.

In this paper we provide further geometric motivation for their definition(s) via the Gelfand-Retakh theory of quasideterminants [4]. In 1997, I. Gelfand and V. Retakh introduced coordinates for Grassmannians over division rings in the hope that *specializations* could provide a universal approach to several

¹ Present address: LaCIM, Université du Québec à Montréal, Case Postale 8888, succursale Centre-ville, Montréal (Québec) H3C 3P8, Canada.
lauve@lacim.uqam.ca

well-known results in noncommutative geometry. This paper realizes that goal for the quantum Grassmannian of Taft and Towber. We interpret our results as evidence that the definitions of quasi-Plücker coordinates are the right ones to provide a noncommutative coordinate geometry—and by extension the “correct” noncommutative algebra—for many noncommutative settings of interest, not just the quantum setting.

This paper begins with a review of the classic Grassmannian and its coordinate algebra. We focus our attention on its description in terms of Plücker coordinates $\{p_I\}$, and Plücker relations. For example, one has the celebrated identity for minors of a 4×2 matrix A :

$$p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0,$$

where p_{ij} represents the determinant of the submatrix of A formed by taking rows i and j and columns 1 and 2.

The intermediate sections introduce quantum and totally noncommutative versions of this story, the latter relying on *quasi-Plücker coordinates*. These are certain ratios of quasideterminants which specialize to ratios of minors in the commutative and quantum settings.

In the final section, we show that the important relations holding among the Plücker coordinates in the classic and quantum setting are consequences of assorted quasideterminantal identities. For example, if we begin with a “generic” 4×2 matrix A and are told that its entries commute with one another, then the identity $(\mathcal{P}_{1,\{2,3\},\{4\}})$ defined in Section 3 reduces to

$$1 = p_{12}p_{32}^{-1}p_{34}p_{14}^{-1} + p_{13}p_{23}^{-1}p_{24}p_{14}^{-1}.$$

Remark. The reader may wish to take a moment to show that the two equations displayed above are equivalent (assuming all symbols p_{ij} are invertible, and $p_{ji} = -p_{ij}$), as it will make some calculations in the sequel more transparent.

In [14] and [15] Lakshmibai and Reshetikhin recall the classic realization of $\mathcal{G}(d, n)$ as a subalgebra (generated by d -minors) of the coordinate algebra for SL_n . With the quantized $\mathcal{O}_{\mathrm{SL}_n}$ and the quantum determinant provided in [20] on hand, the construction of this algebra is straightforward; cf. [16], [3], [12] for modern explorations of its structure. Geometric data appears in the form of representations: they produce its simple modules from the representation theory of $U_q(\mathfrak{sl}_n)$ and use them (along with a modification of Hodge’s “standard monomial theory” [10]) to provide a basis for $\mathcal{G}_q(d, n)$.

Taft and Towber [22] take a more constructive approach. Beginning with a presentation of $\mathcal{G}(d, n)$ by generators and relations, the task was simply to “quantize” this presentation to produce the coordinate ring of a quantum

Grassmannian. The geometric data here is also indirect: following the suggestion of Faddeev, Reshetikhin, and Takhtajan in [20] they verify their algebra is a comodule algebra over the Hopf algebra $\mathcal{O}_{\mathrm{SL}_q(n)}$ just as $\mathcal{G}(d, n)$ is over $\mathcal{O}_{\mathrm{SL}_n}$. They go on to prove that this algebra is the same as the quantum coordinate ring of Lakshmibai and Reshetikhin, strong evidence that indeed this is the “correct” quantum $\mathcal{G}(d, n)$.

The aim of this paper is to give more evidence by realizing the generators and relations of Taft and Towber through more geometric considerations. To this end we use quasideterminants. Other means of attaching geometric data may be found in [19], where Ohn follows the Artin-Tate-van den Bergh approach to noncommutative projective geometry, and in [21], where Škoda uses quasideterminant-theory to provide localizations of the quantum algebras in question.

We fix some notation for the remainder of the paper:

Fix once and for all, positive integers d and n satisfying $d < n$.

By $[n]$ we mean the set $\{1, 2, \dots, n\}$. By $[n]^d$ we mean the set of all d -tuples chosen from $[n]$; while $\binom{[n]}{d}$ denotes the set of all subsets of $[n]$ of size d .

For two integers n, m and two subsets $I \subseteq [n]$ and $J \subseteq [m]$ we define two common matrices associated to an $n \times m$ matrix A : by $A^{I,J}$ we mean the matrix obtained by deleting rows I and columns J from A ; by $A_{I,J}$ we mean the matrix obtained by keeping only rows I and columns J of A . It will be necessary to simplify the above notation in certain cases: when $I = \{i\}$ and $J = \{j\}$, write A^{ij} in place of $A^{I,J}$; when $|I| = d$ and $J = [d]$, write A_I in place of $A_{I,[d]}$.

Given two sets $I, J \subseteq [n]$ with $|I| = d, |J| = e$, write $I|J$ for the tuple $(i_1, \dots, i_d, j_1, \dots, j_e)$.

For $\sigma \in \mathfrak{S}_m$, let $\ell(\sigma) = \ell(\sigma 1, \sigma 2, \dots, \sigma m)$ denote the *length* of the permutation, i.e. the minimal number of adjacent swaps necessary to move $(\sigma 1, \sigma 2, \dots, \sigma m)$ into $(1, 2, \dots, m)$. Extend $\ell(\cdot)$ to elements of $[n]^m$ in the obvious way; we will make frequent use of $\ell(I \setminus \Lambda | \Lambda)$.

By K_q we mean an infinite commutative field K of characteristic 0 with a distinguished element $q \neq 0$ and q not a root of unity.

1 Review of Classical Setting

1.1 Determinants

In this section we work over \mathbb{R} (cf. [24] for a treatment over any commutative ring of characteristic p not dividing $d!$). The determinant of a square matrix A

will be a main organizing tool in what follows. In addition to the well-known alternating property, the determinant has another property the reader should be familiar with:

Proposition 1 (Laplace's Expansion) *Let $A = (a_{ij})_{1 \leq i, j \leq m}$. Suppose that p, p' are fixed positive integers with $p + p' = m$, and that $J = (j_1, \dots, j_m)$ is a fixed derangement of the columns of A . Then*

$$|A| = (-1)^{\ell(J)} \sum (-1)^{-\ell(i_1 \dots i_p i'_1 \dots i'_{p'})} |A_{\{i_1, \dots, i_p\}, \{j_1, \dots, j_p\}}| \cdot |A_{\{i'_1, \dots, i'_{p'}\}, \{j_{p+1}, \dots, j_m\}}|$$

where the sum is over all partitions of $[m]$ into two increasing sets $i_1 < \dots < i_p$ and $i'_1 < \dots < i'_{p'}$.

Typically we take $(j_1, \dots, j_m) = (1, \dots, m)$, so what's written above is the expansion of the determinant down the first p columns of A .

1.2 Grassmannian

First we recall the embedding of the Grassmannian $Gr(d, n)$ into $\mathbb{P}^{\binom{n}{d}-1}$, whose coordinates we will index by the d -subsets of $[n]$. Following [22], we carry out the construction in $V = (\mathbb{R}^n)^*$, not in \mathbb{R}^n .

Given a basis $\mathfrak{B} = \{f_1, \dots, f_n\}$ for $V = \mathbb{R}^n$, we will represent a vector $v \in V^*$ as a n -tuple $(v_1, \dots, v_n)^T$ where $\langle v, f_i \rangle = v_i$. Any d -plane $\Gamma \in Gr(d, n)$ can be represented by any d linearly independent vectors within Γ . We may arrange them as columns in an $n \times d$ matrix via the coordinatization above. It is clear that any two such matrices A, B represent the same Γ if and only if there is an element $g \in GL_d(\mathbb{R})$ satisfying $A = B \cdot g$.

One next forms the map $\eta : Gr(d, n) \rightarrow \mathbb{P}(\mathbb{R}^{\binom{n}{d}})$ as follows. For each Γ , take any matrix representation A and map it to the $\binom{n}{d}$ -tuple of its maximal minors. If A and B as above represent the same Γ , their images will differ only by the scalar $\det g$. Moreover, a matrix A represents an element of $Gr(d, n)$ if and only if at least one maximal minor is nonzero. One concludes that η is well-defined and injective. (This is the Plücker embedding, and we call the coordinates of $p = (p_{\{1, \dots, d\}} : \dots : p_{\{n-d+1, \dots, n\}}) \in \mathbb{P}^{\binom{n}{d}-1}$ the *Plücker coordinates*.)

Proposition 2 *A point $p \in \mathbb{P}^{\binom{n}{d}-1}$ belongs to the image of η if and only if for all $1 \leq r \leq d$ and all choices $I \in \binom{[n]}{d+r}$, $J \in \binom{[n]}{d-r}$, its coordinates satisfy*

$$0 = \sum_{\substack{\Lambda \subseteq I \\ |\Lambda|=r}} (-1)^{\ell(I \setminus \Lambda | \Lambda)} p_{i_1 \dots i_{\lambda_1} \dots i_{\lambda_r} \dots i_{d+r}} p_{i_{\lambda_1} \dots i_{\lambda_r} j_1 \dots j_{d-r}}. \quad (1)$$

These relations take on many equivalent forms, but as written, they shall be called the *Young symmetry relations* $(\mathcal{Y}_{I,J})_{(r)}$. The reader may find a proof in [11], one component of which is the “Basis Theorem” below. Another component is revealed upon inspection of the following determinant:

$$\begin{vmatrix} a_{i_1,1} & \cdots & a_{i_1,d} & a_{i_1,1} & \cdots & a_{i_1,d} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i_{d+r},1} & \cdots & a_{i_{d+r},d} & a_{i_{d+r},d} & \cdots & a_{i_{d+r},d} \\ 0 & \cdots & 0 & a_{j_1,1} & \cdots & a_{j_1,d} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{j_{d-r},1} & \cdots & a_{j_{d-r},d} \end{vmatrix}.$$

Remark. (a) Use a Laplace expansion down the first d columns to see that this determinant takes the form of (1). (b) Subtract the top-left block from the top-right block and discover a hollow matrix, i.e., this determinant is zero.

1.3 Coordinate Algebra

There is one technical detail left unsaid after (1). In the case $I \cap J \neq \emptyset$, the expressions $p_{i_{\lambda_1} \dots i_{\lambda_r} j_1 \dots j_{d-r}}$, will not all correspond to subsets of $[n]$. Moreover, order is important. We need $p_{\{i,j,\dots\}} = -p_{\{j,i,\dots\}}$, etc. We extend the coordinate functions $\{f_I\}$ on the Plücker coordinates to $\{f_I \mid I \in [n]^d\}$ and add the alternating relations (\mathcal{A}_I) :

$$f_I = \begin{cases} \text{sgn}(\sigma) f_J & \text{if } \sigma(I) = J \\ 0 & \text{if two indices are identical.} \end{cases}$$

We are now ready to make the

Definition 3 *The homogeneous coordinate ring of $\mathcal{G}(d,n)$ is the quotient algebra $\mathbb{R}[f_I \mid I \in [n]^d] / (\mathcal{A}_I; \mathcal{Y}_{I,J})$.*

The following theorem suggests we needn’t quotient out by a larger ideal.

Theorem 4 (Basis Theorem [11]) *If F is any homogeneous polynomial in f_I (modulo (\mathcal{A}_I)) such that $F(p) = 0$ for all $p = p(\Gamma) \in Gr(d,n)$, then F is algebraically dependent on the Young symmetry relations; i.e.,*

$$F(p) = \sum_{\substack{I, |I|=d+1 \\ J, |J|=d-1}} H_{I,J}(p) \cdot Y_{I,J}(p),$$

where $Y_{I,J}$ is the homogeneous expression appearing on the right-hand side in $(\mathcal{Y}_{I,J})_{(1)}$ and $H_{I,J}$ is a homogeneous polynomial in the coordinate functions f_I .

Note that, interpreting f_I as $\det(A_I)$, we have that any homogeneous polynomial F of degree m in the f_I satisfies $F(A \cdot g) = F(A)(\det q)^m$ as we expect. In the coming sections, we will mimic the constructions above as best as possible.

2 Quantum Setting

2.1 Quantum Determinants

Before we introduce the q -deformed version of the picture above, we recall several facts about quantum matrices and quantum determinants. The reader may find verification of all unproven statements within this section in [20], [22], or [23].

Definition 5 *An $n \times m$ matrix $X = (x_{ij})$ is called q -generic if its entries satisfy all possible relations of the four types below:*

$$x_{kj}x_{ki} = qx_{ki}x_{kj} \quad (i < j) \tag{2}$$

$$x_{jk}x_{ik} = qx_{ik}x_{jk} \quad (i < j) \tag{3}$$

$$x_{jk}x_{il} = x_{il}x_{jk} \quad (i < j; k < l) \tag{4}$$

$$x_{jl}x_{ik} = x_{ik}x_{jl} + (q - q^{-1})x_{il}x_{jk} \quad (i < j; k < l). \tag{5}$$

Remark. Any submatrix of a q -generic matrix is again q -generic.

We let $M_{n \times m}(q)$ denote the set of all such X . It is a subset of the set of all $n \times m$ matrices with entries in R —the often unenunciated ring of study.

Recall that in commutative linear algebra, one can build the inverse of a matrix A using the determinant:

$$(A^{-1})_{ij} = (\det A)^{-1}(-1)^{j-i} \det A^{ji}. \tag{6}$$

The quantum determinant of a matrix $X = (x_{ij})$ is defined so as to produce the inverse of a q -generic matrix in the same fashion.

Definition 6 *For any square matrix $A = (a_{ij})$ of size n , the quantum determinant $\det_q A = |A|_q$ is defined by*

$$|A|_q = \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)} a_{1\sigma 1} a_{2\sigma 2} \cdots a_{n\sigma n}.$$

Notation. For a subset I of size m , we will frequently use $[I]$ to represent $\det_q(A_{I,\{1,\dots,m\}})$ in order to simplify notation.

Proposition 7 (Properties of quantum matrices) *Let $X = (x_{ij})$ and $Y = (y_{kl})$ be q -generic, with X square and XY defined.*

- (1) *The element $\det_q X$ is central in the algebra $K_q\langle x_{ij} \rangle / (q\text{-generic relations})$.*
- (2) *If X, Y additionally satisfy $x_{ij}y_{kl} = y_{kl}x_{ij} \forall i, j, k, l$ then XY is still q -generic; moreover, if Y is square, $\det_q(XY) = \det_q X \det_q Y$.*
- (3) *The matrix $S(X) := ((-q)^{j-i} \det_q X^{ji})$ satisfies $S(X) \cdot X = X \cdot S(X) = (\det_q X)I_n$, the identity matrix.*

Warning. If $X \in M_{n \times n}(q)$ then $X^{-1} \notin M_{n \times n}(q)$; rather it is a member of $M_{n \times n}(q^{-1})$.

Remark. Item 1 suggests that $(\det_q A_I)(\det_q A_J) = (\det_q A_J)(\det_q A_I)$ whenever $J \subseteq I$. This will be quite useful in the sequel.

For all $1 \leq m \leq n$, define $GL_q(m)$ to be $GL_m(R) \cap M_{m \times m}(q)$ —the q -generic matrices which are invertible over R . There is not a true group or semigroup structure on this set, e.g. if X is 2×2 q -generic, then X^2 is not. However, Proposition 7 suggests that a trace of the desired structure remains: $X \cdot Y \in GL_q(m)$ when the coordinates of X commute with those of Y .

There are two more properties of \det_q which we will need. The first is the “ q -alternating” property of Taft and Towber [22].

Theorem 8 *Suppose X is an $n \times n$ q -generic matrix, and A is built by choosing rows i_1, \dots, i_n (not necessarily distinct) from X . Then*

$$\det_q A = \begin{cases} (-q)^{-\ell(i_1 \dots i_n)} \det_q X & \text{if all rows are distinct} \\ 0 & \text{otherwise} \end{cases}. \quad (7)$$

The second property is that often two quantum minors “ q -commute:”

Definition 9 *Two quantum minors $[I]$ and $[J]$ of a q -generic matrix X are said to q -commute if there is an integer b so that $[J][I] = q^b[I][J]$.*

For example, we have this

Proposition 10 *Suppose $i, j \in [n]$ and $M \subset [n]$, with $|M| < n$ and $i < j$. Then the quantum minors $[i \cup M]$ and $[j \cup M]$ satisfy*

$$[j \cup M][i \cup M] = q[i \cup M][j \cup M]. \quad (8)$$

LeClerc and Zelevensky actually prove a much stronger result in [16]—giving necessary and sufficient conditions on subsets I, J in order that $[I]$ and $[J]$ q -commute. However, their proof involves machinery from [22] which we wish to avoid. We present a simple proof of this weak- q -commuting property in Section 4.

2.2 Quantum Space

We are now ready to q -deform the picture in Section 1.2. We move from a vector space over \mathbb{R} to n -dimensional “quantum space” V_q over the field K_q . We begin by considering a vector space D^n with basis $\mathfrak{B} = \{f_1, \dots, f_n\}$, where D is some (unspecified) division algebra over K_q . We take V as the left D -vector space $V = (D^n)^* = \text{Hom}_D(D^n, D)$; again we build coordinates for vectors $v \in V$ from their behavior on \mathfrak{B} .

We will call a **point** in V q -generic if its coordinates satisfy $v_j v_i = q v_i v_j$ ($\forall j > i$). These are the points we wish to study; we call this set V_q . **WARNING:** this is *not* a vector space over K_q (or D) as it is not closed under addition. However, the K_q -action inherited from D (it being a K_q -algebra) is well-defined. For if $\alpha \in K_q$, and $v = (v_1, \dots, v_n)^T \in V_q$, then $\alpha \cdot v \in V_q$ as well (e.g. $(\alpha v_2)(\alpha v_1) = \alpha v_2 v_1 \alpha = \alpha q v_1 v_2 \alpha = q(\alpha v_1)(\alpha v_2)$).

We will call a d -dimensional **subspace** W of V q -generic if there is a linearly independent set $\{v^1, \dots, v^d\} \in V_q \cap W$ so that $A = [v^1 | \dots | v^d] \in M_{n \times d}(q)$. As in the commutative case, A will represent a point in $Gr_q(d, n)$.

2.3 Quantum Grassmannian

Finally, we define $Gr_q(d, n)$ as a quotient of $M_{n \times d}(q)$. We take $A \sim B$ if there is a finite sequence of matrices $\{X_i\}_{1 \leq i \leq t}$ chosen from $GL_q(d) \cup GL_{q^{-1}}(d)$ satisfying: (i) $(A \cdot X_1 \cdots X_{i-1}) \cdot X_i$ is q -generic for all $1 \leq i \leq t$; (ii) $B = AX_1 \cdots X_t$; (iii) $\det_q B_I = (\det_q A_I) \cdot (\det_q(X_1 \cdots X_t))$ for all $I \in \binom{[n]}{d}$.

Definition 11 *We have defined $Gr_q(d, n)$ above in terms of matrices. We would like to have a coordinates version as in the commutative case. We identify a point Γ_q in the quantum Grassmannian $Gr_q(d, n)$ with the set of maximal quantum minors of $A(\Gamma_q)$ —its quantum Plücker coordinates.*

Remark. Condition (iii) above is fairly restrictive, but it allows us to safely identify two sets of coordinates up to a scalar. We will see shortly that even this is not restrictive enough to completely mimic the classical setting.

From Section 2.1 it is clear that a coordinate $[I]$ of Γ_q is q -alternating in I . The coordinates also satisfy a quantized version of the Young symmetry relations².

2.4 Quantized Coordinate Algebra

Following the classical picture outlined above, we make the

Definition 12 Put $\mathcal{G}_q(d, n) = K_q \langle f_I \mid I \in [n]^d \rangle / (\mathcal{A}_I; \mathcal{Y}_{I,J})$ where (\mathcal{A}_I) , and $(\mathcal{Y}_{I,J})$ are now appropriate quantized versions of those from Section 1.2:

the alternating relations (\mathcal{A}_I)

$$f_I = \begin{cases} (-q)^{-\ell(I)} f_{\sigma(I)} & \text{if } \sigma \text{ orders the entries of } I \\ 0 & \text{if two indices are identical.} \end{cases}$$

the Young symmetry relations $(\mathcal{Y}_{I,J})$

$$0 = \sum_{\substack{\Lambda \subseteq I \\ |\Lambda|=r}} (-q)^{-\ell(I \setminus \Lambda | \Lambda)} f_{I \setminus \Lambda} f_{\Lambda | J} \quad (9)$$

for all $1 \leq r \leq d$, $I \in \binom{[n]}{d+r}$, and $J \in \binom{[n]}{d-r}$.

In [22] Taft and Towber give this same definition for the homogeneous coordinate ring of the quantum Grassmannian. They go on to prove a quantized version of the basis theorem:

the subalgebra inside $K_q \langle x_{ij} \mid q\text{-relations} \rangle$ generated by $\{[I]\}_{|I|=d}$ is isomorphic to $\mathcal{G}_q(d, n)$.

So not only are the f_I well-defined functions on the points $\Gamma_q \in Gr_q(d, n)$, it would seem $\mathcal{G}_q(d, n)$ is the biggest quotient algebra of $K_q \langle f_I \rangle$ with this property.

Remark. Note that when we interpret f_I as $\det_q(A_I)$ we have $f_I(A \cdot g) = f_I(A) \det_q(g)$ whenever $A \cdot g \sim A$. Suppose we additionally know that the entries of g commute with those of A , then if F is any homogeneous polynomial in $\mathcal{G}_q(d, n)$ of degree m we have $F(A \cdot g) = F(A) \cdot (\det_q(g))^m$. This seems to be as close to the classical case as we can come...and not even this is true

² For completeness, it should be noted that the proof of this fact which appears in [22] uses a q -Laplace expansion in much the same spirit as the classic Laplace expansion was used in the discussion following equation (1).

if we do not add this assumption about g . However, we may make a more satisfactory comparison to the classical case when we consider “homogeneous degree zero” rational functions in the $\{f_I\}$ (cf. Proposition 32).

The algebra $\mathcal{G}_q(d, n)$ has been well studied since its introduction (cf. [3, 9, 12, 22]). In this paper we concentrate on $Gr_q(d, n)$ itself.

2.5 Young Symmetry Relations, Simplified

In the classical construction of $\mathcal{G}(d, n)$ it is known that all relations of the type in (1) with $r > 1$ are direct consequences of those with $r = 1$ (cf. [11] and [24]). The proofs published there rely heavily on the commutativity of the Plücker coordinates $\{p_I\}$. What follows is a proof of the same fact for quantum Plücker coordinates. In addition to giving a new proof for the classical case (set $q = 1$), it represents the key lemma for what follows in *Section 4*.

Notation. Given an ordered set L of size n and its r -th element l_r , let $L_{(r)}$ denote $L \setminus \{l_r\}$. In the event that $l_r \notin L$ we interpret $L_{(r)}$ as simply a reminder of this fact (i.e. $L_{(r)} = L$). For two subsets $A = \{a_1, \dots, a_s\}$ and $B = \{b_1, \dots, b_t\}$ of $\{1, \dots, n\}$, let $[A|B]$ denote $\det_q T_{\{a_1, \dots, a_s, b_1, \dots, b_t\}, \{1, \dots, s+t\}}$ for some q -generic matrix T .

Proposition 13 *Let I, J be ordered subsets of $[n]$ with respective sizes $d + r$ and $d - r$ ($1 \leq r \leq d \leq n$). Then $(\mathcal{Y}_{I,J})_{(r)}$ can be written in terms of relations of type $(\mathcal{Y}_{L,M})_{(r-1)}$. Specifically,*

$$\begin{aligned} \sum_{s=1}^{d+r} (-q)^{2(r-1)-\ell(I_{(s)}|i_s)} \sum_{\substack{\Lambda_{(s)} \subset I_{(s)} \\ |\Lambda_{(s)}|=r-1}} (-q)^{-\ell(I_{(s)} \setminus \Lambda_{(s)} | \Lambda_{(s)})} [I_{(s)} \setminus \Lambda_{(s)}] [\Lambda_{(s)} | i_s | J] \\ = \left(\sum_{t=0}^{r-1} (-q)^{2t} \right) \sum_{\substack{\Lambda \subset I \\ |\Lambda|=r}} (-q)^{-\ell(I \setminus \Lambda | \Lambda)} [I \setminus \Lambda] [\Lambda | J]. \end{aligned}$$

PROOF. We simply take an arbitrary Λ and compare the coefficients on the left- and right-hand sides of the monomial $[I \setminus \Lambda] [\Lambda | J]$.

left-hand side:

$$\begin{aligned} \sum_{i_s \in \Lambda} (-q)^{2(r-1)-\ell(I_{(s)}|i_s)} (-q)^{-\ell(I_{(s)} \setminus \Lambda_{(s)} | \Lambda_{(s)})} [I \setminus \Lambda] [\Lambda_{(s)} | i_s | J] \\ = \left(\sum_{i_s \in \Lambda} (-q)^{2(r-1)-\ell(I_{(s)}|i_s)-\ell(I_{(s)} \setminus \Lambda_{(s)} | \Lambda_{(s)})-\ell(\Lambda_{(s)}|i_s)} \right) [I \setminus \Lambda] [\Lambda | J] \end{aligned}$$

right-hand side:

$$\left(\sum_{t=0}^{r-1} (-q)^{2t - \ell(I \setminus \Lambda | \Lambda)} \right) [I \setminus \Lambda][\Lambda | J].$$

Multiplying both sides by $(-q)^{+\ell(I \setminus \Lambda | \Lambda)}$ and using $\ell(I \setminus \Lambda | \Lambda) = \ell(I \setminus \Lambda | \Lambda_{(s)}) + \ell(I_{(s)} | i_s) - \ell(\Lambda_{(s)} | i_s)$, we are left with showing

$$\sum_{s=0}^{r-1} (-q)^{2(r-1) - 2\ell(\Lambda_{(s)} | i_s)} = \sum_{t=0}^{r-1} (-q)^{2t}.$$

But $(r-1) - \ell(\Lambda_{(s)} | i_s)$ is exactly s . \square

Repeated application of this reduction proves the following important modification to the quantized basis theorem.

Corollary 14 *Equation (9) in the definition of the $\mathcal{G}_q(d, n)$ can be replaced with an abbreviated version—taking only $r = 1$.*

Remark. (a) Note that this proof fails to work if q^2 is an r -th root of unity. In the case $q = 1$ it additionally fails if the characteristic of the field is r . Thus there is no improvement to the situation addressed in [24] in the commutative case. (b) The lemma was proven for $(|J|, |I|) = (d-r, d+r)$, but a generalization to the setting $(|J|, |I|) = (s-r, t+r)$ with $0 \leq r \leq s \leq t \leq d$ is immediate. This extended identity will be utilized in a later paper when we address noncommutative flags.

3 Generic Setting

3.1 Quasideterminants

Gelfand and Retakh suggest that the quasideterminant should be a main organizing tool in noncommutative mathematics; and indeed it has already provided explicit formulas to a variety of noncommutative problems (finding Casimir elements [8], [18] and factoring noncommutative polynomials [2],[7] are two notable examples). The results of this paper provide further support for this suggestion.

The computations in this subsection will be done in the free skew field $K\langle a_{ij} \rangle$ (cf. [1]) built on the entries of a matrix A with distinct noncommuting indeterminates. As the definition will make clear, if we instead work with A over an arbitrary noncommutative ring R some quasideterminants may not be defined. A careful study of [1] reveals that quasideterminants are elements of

certain localizations of R . The reader will find a more thorough treatment of the quasideterminant and its properties, including some of the proofs omitted below, in [6] and [13].

Definition 15 (Quasideterminant, I) *An $n \times n$ matrix A has in general n^2 quasideterminants, one for each position in A . The (ij) -quasideterminant is defined as follows:*

$$|A|_{ij} = a_{ij} - \sum_{r \neq i, s \neq j} a_{is} \left(|A^{ij}|_{rs} \right)^{-1} a_{rj}.$$

One may use this definition and (6) to easily conclude that in the commutative case, the quasideterminant specializes to the ratio of two determinants:

$$|A|_{ij} = (-1)^{i+j} (\det A) / (\det A^{ij}).$$

Notation. It will be convenient to denote the (ij) -quasideterminant in another form:

$$\left| \begin{array}{ccc} \vdots & & \\ \cdots & a_{ij} & \cdots \\ \vdots & & \end{array} \right|_{ij} = \left| \begin{array}{ccc} \vdots & & \\ \cdots & \boxed{a_{ij}} & \cdots \\ \vdots & & \end{array} \right|.$$

There is an alternate definition which we will also have occasion to use. Let ξ be the i -th row of A with the j -th coordinate deleted; and let ζ be the j -th column of A with the i -th coordinate deleted.

Definition 16 (Quasideterminant, II) *For A, ξ, ζ as above, the (ij) -quasideterminant is defined as follows:*

$$|A|_{ij} = a_{ij} - \xi (A^{ij})^{-1} \zeta.$$

In attempting to make these two definitions agree, one stumbles upon the first fundamental fact about quasideterminants,

$$(|A|_{ij})^{-1} = (A^{-1})_{ji}, \tag{10}$$

when the right-hand side is defined and not equal to zero.

The quasideterminant is extremely well-behaved for being a non-commutative determinant (or rather ratio of two). Consider its behavior under elementary transformations of columns.

Proposition 17 *Let $A = (a_{ij})$ be a square matrix.*

- (Column Permutations) Suppose $\tau \in S_n$ and P_τ is the associated (column) permutation matrix. Then $|AP_\tau|_{i,\tau j} = |A|_{i,j}$.
- (Rescaling Columns) Let B be the matrix obtained from A by multiplying its r th column by ρ on the right. Then

$$|B|_{ij} = \begin{cases} |A|_{ij} \rho & \text{if } j = r \\ |A|_{ij} & \text{if } j \neq r \text{ and } \rho \text{ is invertible.} \end{cases}$$

- (Adding to Columns) Let B be the matrix obtained from A by adding column r (multiplied on the right by a scalar ρ) to column s . Then $|B|_{ij} = |A|_{ij}$ if $j \neq r$.

See [6] for more details (and for row versions of all the properties in this subsection). With these properties, we may easily deduce the following

Proposition 18 *If A is a square matrix and column s of A is a right-linear combination of the other columns, then $|A|_{rs} = 0$ (whenever it is defined).*

Remark. A row version of this is true as well, and will be used below.

PROOF. Through a sequence of steps $A = A(0), \dots, A(m) = B$, column-reduce A to a matrix B : $\text{col}_s(B) = 0$; $\text{col}_j(B) = \text{col}_j(A)$ ($j \neq s$). Then Proposition 17 above indicates

$$|A|_{rs} = |A(i)|_{rs} \quad (\forall 1 \leq i \leq m).$$

Finally, use the second definition of quasideterminant to conclude that $|B|_{rs}$ is indeed zero. \square

Proposition 19 (Column Homological Relations) *Let $A = (a_{ij})$ be a square matrix. Then*

$$-|A^{kj}|_{il}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kl}^{-1} \cdot |A|_{kj} \quad (\forall l \neq j).$$

We will also find a use for the following identity of Krob and LeClerc, which gives a *one-column* Laplace expansion of the quasideterminant.

Proposition 20 *For $A = (a_{ij})$, the (ij) -quasideterminant has the following expansion:*

$$|A|_{rs} = a_{rs} - \sum_{i \neq r} |A^{is}|_{rl} \cdot |A^{rs}|_{il}^{-1} \cdot a_{is} \quad (\forall l \neq s). \quad (11)$$

PROOF. From (10) and the previous proposition we have

$$\begin{aligned}
1 &= \sum_{i=1}^n |A|_{is}^{-1} \cdot a_{is} \\
|A|_{rs} &= a_{rs} + \sum_{i \neq r} |A|_{rs} \cdot |A|_{is}^{-1} \cdot a_{is} \\
|A|_{rs} &= a_{rs} - \sum_{i \neq r} |A^{is}|_{rl} \cdot |A^{rs}|_{il}^{-1} \cdot a_{is}. \quad \square
\end{aligned}$$

3.2 Noncommutative Plücker Coordinates

We may use the quasideterminant to build noncommutative Plücker coordinates. One cannot simply replace the determinants appearing earlier with quasideterminants, because the latter are not invariant (up to scalar) under GL_d action. In [4,5], Gelfand and Retakh give evidence that certain ratios of quasideterminants are the proper substitute.

Definition 21 (Quasi-Plücker Coordinates) *Let A be a matrix of size $n \times d$ ($n \geq d$). Let M be a subset of $[n]$ of cardinality $d-1$, and suppose $i, j \in [n]$ with $i \notin M$. A (right-) quasi-Plücker coordinate for A will be defined as $r_{ji}^M(A) := |A_{j \cup M}|_{js} \cdot |A_{i \cup M}|_{is}^{-1}$ (for any $1 \leq s \leq d$).*

Proposition 22 (Compelling Properties) *For A, M, i , and j , as above, the quasi-Plücker coordinates satisfy the following:*

- $r_{ji}^M(A)$ does not depend on s
- $r_{ji}^M(A \cdot g) = r_{ji}^M(A)$ for any $g \in \mathrm{GL}_d$

If we associate a point Γ in a noncommutative Grassmannian—i.e. a submodule of $V_D = (D^n)^*$ isomorphic to D^d for some division ring D —to an $n \times d$ matrix A in a manner similar to what has come before, we might take the quasi-Plücker coordinates of Γ to be the $n^2 \binom{n-1}{d-1}$ “minors” r_{ji}^M .

Additional nice properties of the r_{ji}^M are worth mentioning.

Proposition 23 *For A, M , and i as above the following also hold:*

- $r_{ji}^M(A)$ does not depend on the ordering of M
- $r_{ji}^M(A) = \begin{cases} 0 & \text{if } j \in M \\ 1 & \text{if } j = i \end{cases}$
- $r_{ji}^M r_{il}^M = r_{jl}^M \quad (l \notin M)$
- $r_{ij}^{M \cup l} r_{jl}^{M \cup i} r_{li}^{M \cup j} = -1 \quad (j, l \notin M)$

3.3 Noncommutative Grassmannian

The fundamental identity holding among the coordinates appears below. It was first observed in [5]. We call this identity the “quasi-Plücker relations.” It will allow us to describe Grassmannians and Grassmann algebras in a manner similar to that used in Section 2.3.

Proposition 24 (Quasi-Plücker Relations) *Let A be an $n \times d$ matrix ($n \geq d$). Then for all subsets $\{i\}$, $M = \{m_2, \dots, m_d\}$, $L = \{l_1, \dots, l_d\}$ chosen from $\{1, \dots, n\}$ with $i \notin M$, we have*

$$(\mathcal{P}_{i,L,M}) : \sum_{j \in L} r_{ij}^{L \setminus j}(A) \cdot r_{ji}^M(A) = 1. \quad (12)$$

PROOF. Using the definition of the quasi-Plücker coordinates, we show that

$$1 = \sum_{j \in L} |A_{i \cup (L \setminus j)}|_{ir} \cdot |A_{j \cup (L \setminus j)}|_{jr}^{-1} \cdot |A_{j \cup M}|_{js} \cdot |A_{i \cup M}|_{is}^{-1} \quad (\forall 1 \leq r, s \leq d).$$

Let $\xi = (\xi_0, \dots, \xi_t)^T$ be the column vector defined as follows:

$$\xi_j = \begin{cases} |A_{i \cup M}|_{ir} & \text{if } j = 0 \\ |A_{l_j \cup M}|_{l_j r} & \text{otherwise} \end{cases}.$$

Let B be the matrix $A_{\{i \cup L\}, \{1, \dots, d\}}$ and form the augmented matrix $C = [\xi | B]$.

Lemma 25 *The matrix C is non-invertible, in particular $|C|_{11} = 0$.*

Using the second definition of quasideterminants, we first notice that

$$\begin{aligned} \xi_0 &= \begin{vmatrix} a_{i1} & \cdots & \boxed{a_{ir}} & \cdots & a_{id} \\ a_{m_2 1} & \cdots & a_{m_2 r} & \cdots & a_{m_2 d} \\ \vdots & & \vdots & & \vdots \\ a_{m_d 1} & \cdots & a_{m_d r} & \cdots & a_{m_d d} \end{vmatrix} \\ &= a_{ir} - \sum_{s \neq r} a_{is} \sum_{t=2}^d \left| (A_{i \cup M})^{ir} \right|_{m_t s}^{-1} \cdot a_{m_t s}. \end{aligned}$$

Computing all of its coordinates at once, we have

$$\begin{aligned}
\xi &= \text{col}_r(B) - \text{col}_1(B) \cdot \sum_{t=2}^d \left| (A_{i \cup M})^{ir} \right|_{m_t 1}^{-1} \cdot a_{m_t 1} - \cdots \\
&\quad - \text{col}_d(B) \cdot \sum_{t=2}^d \left| (A_{i \cup M})^{ir} \right|_{m_t d}^{-1} \cdot a_{m_t d} \\
&= \sum_{j=1}^d \text{col}_j(B) \cdot \lambda_j.
\end{aligned}$$

Hence the first column is a right-linear combination of the latter columns. In particular, Proposition 18 implies that $|C|_{11} = 0$. \diamond

We next employ (11) to $|C|_{11}$ to get the final result:

$$\begin{aligned}
0 &= \xi_0 - \sum_{j=1}^d |A_{i \cup (L \setminus j)}|_{ir} \cdot |A_{j \cup (L \setminus j)}|_{jr}^{-1} \cdot \xi_j \quad (\forall r) \\
1 &= \sum_{j=1}^d |A_{i \cup (L \setminus j)}|_{ir} \cdot |A_{j \cup (L \setminus j)}|_{jr}^{-1} \cdot |A_{j \cup M}|_{js} \cdot |A_{i \cup M}|_{is}^{-1} \quad (\forall r, s) \\
1 &= \sum_{j \in L} r_{ij}^{L \setminus j} \cdot r_{ji}^M. \quad \square
\end{aligned}$$

Remark. The proof appearing above is new and has an obvious generalization: we only need $0 \leq |M| \leq |L| - 1 \leq d$ to make the proof work. We will explore this extended identity in a later paper when we address noncommutative flag coordinates. We identify a point Γ in the Grassmannian with its collection of quasi-Plücker coordinates $\{r_{ji}^M\}$.

3.4 Toward a Coordinate Algebra

One would like a definition of the following sort: the homogeneous coordinate ring of the Grassmannian in the noncommutative setting is the algebra with generators r_{ij}^M and relations all those described above in Proposition 23 and (12). However, as all of the symbols are invertible, it seems an algebra of rational functions is more appropriate. In this setting, we have the following theorem.

Theorem 26 *Let $A = (a_{ij})$ be a $n \times d$ matrix with formal entries and let $f(a_{ij})$ be a rational function over the free skew-field D generated by the a_{ij} . Suppose f is invariant under all invertible transformations $A \mapsto A \cdot g$, ($g \in \text{GL}_d(D)$). Then f is a rational function of the quasi-Plücker coordinates $r_{ij}^M(A)$.*

PROOF. Let $B = A_{\{1,\dots,d\},\{1,\dots,d\}}$ and consider the matrix $C = A \cdot B^{-1}$. Then $f(A) = f(C)$, and Gelfand and Retakh have shown that

$$(C)_{ij} = \begin{cases} \delta_{ij} & j \leq d \\ r_{ij}^{\{1,\dots,\hat{i},\dots,d\}}(A) & j > d \end{cases}. \quad \square$$

Finally, we would like a version of the basis theorem to be true, e.g. *if f is a rational function in the coordinates r_{ij}^M with $f(A) = 0$, then $f = f(\mathcal{P}_{i,L,M})$ is zero because it can be written in terms of the quasi-Plücker relations.* This may be true, but any such theorem is still pending.

4 Quasi \leadsto Quantum

In this final section, we return our focus to quantum things (similar results being obtainable for the commutative case via further specialization $q \rightarrow 1$).

4.1 Coordinates

Given a q -generic matrix X , we have seen that the (ij) -th entry of X^{-1} is $(\det_q X)^{-1}(-q)^{j-i} \det_q(X^{ji})$. We have also related the (ji) -th quasideterminant of X to the (ij) -th entry of X^{-1} . A brief study of this relation yields the following essential formula³, first introduced in [4]:

$$\det_q X = (-q)^{\ell(i_1 \cdots i_n) - \ell(j_1 \cdots j_n)} \left| X \right|_{i_1, j_1} \left| X^{i_1 j_1} \right|_{i_2, j_2} \cdots \left| x_{i_n j_n} \right|_{i_n, j_n}; \quad (13)$$

moreover, all of the terms on the right-hand side commute with each other Proposition 7. We may extend (13) to give quantum determinant expansions for certain matrices associated to X .

Proposition 27 *Let A be a square matrix, with rows i_1, \dots, i_m not necessarily ordered (and not necessarily distinct) chosen from the rows of a q -generic matrix X . Then*

$$\det_q A = \left| A \right|_{i_1 1} \left| A^{i_1 1} \right|_{i_2 2} \left| A^{i_1 i_2, 12} \right|_{i_3 3} \cdots a_{i_m m}. \quad (14)$$

³ This formula is not unique to quantum determinants. Many of the famous non-commutative determinants exhibit this property in some form or another (cf. [6]).

PROOF. If $\det_q A = 0$, then the (i_s) -th row is the same as some row i_t ($s < t$) by (7)). In this case, $\left| A^{\{i_1 \cdots i_{s-1}\}, \{i_1 \cdots i_{s-1}\}} \right|_{i_s, i_s} = 0$ by Proposition 18 (row version).

Otherwise, let $\sigma(j) = i_j$ for $j = 1 \dots n$ and use equation (7) to rewrite (13) as follows:

$$\begin{aligned} (-q)^{\ell(\sigma)} \det_q A &= \det_q X \\ &= (-q)^{\ell(\sigma)} \left| X \right|_{\sigma 1, 1} \left| X^{\sigma 1, 1} \right|_{\sigma 2, 2} \cdots \left| x_{\sigma n, n} \right|_{\sigma n, n} \\ &= (-q)^{\ell(\sigma)} \left| \sigma^{-1} X \right|_{11} \left| (\sigma^{-1} X)^{11} \right|_{22} \cdots \left| (\sigma^{-1} X)_{nn} \right|_{nn} \\ &= (-q)^{\ell(\sigma)} \left| A \right|_{11} \left| A^{11} \right|_{22} \cdots \left| a_{nn} \right|_{nn}, \end{aligned}$$

where σ^{-1} acts on X by row permutations. \square

Notation. For a subset I of size m , we will have occasion to use $\left| i_1 \cdots \boxed{i_s} \cdots i_m \right|$ for the $(i_s 1)$ -quasideterminant of the matrix $A_{I, \{1, \dots, m\}}$. For example, if B is a 2×2 matrix, with rows i and j taken from some larger matrix A , then:

$$\left| B \right|_{j1} = \left| \begin{array}{cc} a_{i1} & a_{i2} \\ \boxed{a_{j1}} & a_{j2} \end{array} \right| = \left| i \boxed{j} \right|.$$

Using this notation—together with the shorthand notation for $\det_q(A_I)$ described above—the reader may check that the following identities hold:

- $\left| \boxed{i_1} \cdots i_d \right| = [i_1 \cdots i_d] \det_q \left(A_{\{i_2, \dots, i_d\}, \{2, \dots, d\}} \right)^{-1}$
- $\left| \boxed{i} m_2 \cdots m_d \right| \left| \boxed{j} m_2 \cdots m_d \right|^{-1} = [i m_2 \cdots m_d] [j m_2 \cdots m_d]^{-1}$

PROOF, Proposition 10. Consider the following column homological relation for the q -generic matrix $A_{i \cup j \cup M}$:

$$\begin{aligned} - \left| A^{j1} \right|_{i2}^{-1} \cdot \left| A \right|_{i1} &= \left| A^{i1} \right|_{j2}^{-1} \cdot \left| A \right|_{j1} \\ - \left| A^{i1} \right|_{j2} \cdot \left| A^{j1} \right|_{i2}^{-1} &= \left| A \right|_{j1} \cdot \left| A \right|_{i1}^{-1}. \end{aligned}$$

We apply the simple identities above, the q -alternating property, and Proposition 7(1) to finish the proof.

Left-hand side:

$$\begin{aligned} -|A^{i1}|_{j2} \cdot |A^{j1}|_{i2}^{-1} &= -\left| \begin{bmatrix} j \\ m_2 \cdots m_d \end{bmatrix} \right| \left| \begin{bmatrix} i \\ m_2 \cdots m_d \end{bmatrix} \right|^{-1} \\ &= -[jm_2 \cdots m_d][im_2 \cdots m_d]^{-1}, \end{aligned}$$

using the identities above starting from column 2 of the original matrix A .

Right-hand side:

$$\begin{aligned} |A|_{j1} \cdot |A|_{i1}^{-1} &= \left| \begin{bmatrix} j \\ m_2 \cdots m_d \end{bmatrix} \right| \left| \begin{bmatrix} i \\ m_2 \cdots m_d \end{bmatrix} \right|^{-1} \\ &= \left([jm_2 \cdots m_d][im_2 \cdots m_d]^{-1} \right) \cdot \left([im_2 \cdots m_d][jm_2 \cdots m_d]^{-1} \right)^{-1} \\ &= [im_2 \cdots m_d]^{-1} [jm_2 \cdots m_d] [im_2 \cdots m_d]^{-1} [jm_2 \cdots m_d] \\ &= -q^{\pm 1} [im_2 \cdots m_d]^{-1} [jm_2 \cdots m_d], \end{aligned}$$

where the power of $-q$ depends on whether $i < j$ or $i > j$. Note the heavy reliance on the centrality of quantum determinants, Proposition 7(1). The result now follows by clearing denominators. \square

4.2 Grassmannians

In this section we prove the main result of this paper, the quantized coordinate algebra $\mathcal{G}_q(d, n)$ of Taft and Towber results from specializing the geometry of the generic Grassmannian.

Theorem 28 (Quasi-Specialization) *Let A be an $n \times d$ q -generic matrix representing a point Γ_q in the quantum Grassmannian $Gr_q(d, n)$. Then all the relations among the coordinates $\{[I] \mid I \in \binom{[n]}{d}\}$ of Γ_q are consequences of the coordinate-relations for its quasi-Plücker coordinates $\{r_{ji}^M(A)\}$.*

Alternatively, beginning with an $n \times d$ matrix A of indeterminants over the free skew field built on the $\{a_{ij}\}$, all relations of the form $(\mathcal{Y}_{I,J}^*)$ are direct consequences of adding q -genericity to the quasi-Plücker relations $(\mathcal{P}_{i,L,M})$ already holding for A .

PROOF. Along with q -genericity we add its easy consequences—the q -alternating and (weak) q -commuting properties of equations (7) and (8).

We have as our target $(\mathcal{Y}_{IJ})_{(r)}$. By Corollary 14, we may assume $r = 1$; so let $I = \{i_1, \dots, i_{d+1}\}$ and $J = \{j_1, \dots, j_{d-1}\}$. Starting from the relation $(\mathcal{P}_{i_1, J_{(1)}, J})$

we have:

$$\begin{aligned}
1 &= \sum_{j \in L} r_{ij}^{L \setminus j} \cdot r_{ji}^M \\
1 &= \sum_{j \in L} \left| \boxed{i} L \setminus j \right| \left| \boxed{j} L \setminus j \right|^{-1} \cdot \left| \boxed{j} M \right| \left| \boxed{i} M \right|^{-1} \\
1 &= \sum_{2 \leq \lambda \leq d+1} \left| \boxed{i_1} I_{(1)} \setminus i_\lambda \right| \left| \boxed{i_\lambda} I_{(1)} \setminus i_\lambda \right|^{-1} \cdot \left| \boxed{i_\lambda} J \right| \left| \boxed{i_1} J \right|^{-1} \\
1 &= \sum_{2 \leq \lambda \leq d+1} \left[i_1 I_{(1)} \setminus i_\lambda \right] \left[i_\lambda I_{(1)} \setminus i_\lambda \right]^{-1} \cdot \left[i_\lambda J \right] \left[i_1 J \right]^{-1} \\
\left[i_1 J \right] &= \sum_{2 \leq \lambda \leq d+1} q \left[i_\lambda I_{(1)} \setminus i_\lambda \right]^{-1} \left[i_1 I_{(1)} \setminus i_\lambda \right] \left[i_\lambda J \right] \\
\left[i_1 J \right] &= \sum_{2 \leq \lambda \leq d+1} q(-q)^{+\ell(i_\lambda | I_{(1)} \setminus i_\lambda)} \left[I_{(1)} \right]^{-1} \left[i_1 I_{(1)} \setminus i_\lambda \right] \left[i_\lambda J \right] \\
\left[I_{(1)} \right] \left[i_1 J \right] &= \sum_{2 \leq \lambda \leq t+r} q(-q)^{+\ell(i_\lambda | I_{(1)} \setminus i_\lambda)} \left[i_1 I_{(1)} \setminus i_\lambda \right] \left[i_\lambda J \right] \\
\left[I_{(1)} \right] \left[i_1 J \right] &= - \sum_{2 \leq \lambda \leq d+1} (-q)^{+\ell(i_\lambda | I \setminus i_\lambda)} \left[I \setminus i_\lambda \right] \left[i_\lambda J \right]
\end{aligned}$$

$$\begin{aligned}
0 &= (-q)^{-\ell(I_{(1)} | i_1)} \left[I_{(1)} \right] \left[i_1 J \right] + \sum_{2 \leq \lambda \leq d+1} (-q)^{+\ell(i_\lambda | I \setminus i_\lambda)} (-q)^{-\ell(I_{(1)} | i_1)} \left[I \setminus i_\lambda \right] \left[i_\lambda J \right] \\
0 &= (-q)^{-\ell(I_{(1)} | i_1)} \left[I_{(1)} \right] \left[i_1 J \right] + \sum_{2 \leq \lambda \leq d+1} (-q)^{-\ell(I_{(\lambda)} | i_\lambda)} \left[I_{(\lambda)} \right] \left[i_\lambda J \right] \\
0 &= \sum_{1 \leq \lambda \leq d+1} (-q)^{-\ell(I_{(\lambda)} | i_\lambda)} \left[I_{(\lambda)} \right] \left[i_\lambda J \right].
\end{aligned}$$

This is exactly the targeted (\mathcal{Y}_{IJ}) . Now, we implicitly began with the assumption $i = i_1 \notin J$, but any choice from $I \setminus J$ could have been made for i . Finally, if $I \setminus J = \emptyset$, then (9) reads $0 = 0$ by the q -alternating property. \square

4.3 Coordinate Algebras

We conclude this section with the introduction of a natural algebra of functions on the quantum Grassmannian. This algebra is invariant under the relation \sim introduced in Section 2.3. Moreover, its elements F are identically zero on $Gr_q(d, n)$ only if they are zero for quasi-Plücker reasons.

In [12] we learn that $\mathcal{G}_q(d, n)$ is a noetherian domain, and as such has a (right, Ore) skew-field of fractions \mathcal{D} . Namely, every element of \mathcal{D} can be written as GH^{-1} with $G, H \in \mathcal{G}_q(d, n)$. This field is too big to be an appropriate field of fractions for $Gr_q(d, n)$; we look for the \sim -invariant functions within \mathcal{D} .

Proposition 29 *Let R be a noetherian domain with right field of fractions D . If R is graded, then the subset $D_0 = \{gh^{-1} \in R \mid g, h \text{ are homogeneous of the same degree}\}$ is a well-defined subfield of D .*

PROOF. Given ef^{-1} and gh^{-1} in D_0 , we may add and multiply these two fractions together by the Ore conditions in D :

- (+) : We know $\exists u, v \in R$ with $fu = hv$. So we may write $ef^{-1} + gh^{-1} = (eu)(fu)^{-1} + (gv)(hv)^{-1} = (eu + gv)(hv)^{-1}$.
 (\times) : we know $\exists u', v' \in R$ with $fu' = gv'$. So we may write $ef^{-1}gh^{-1} = (eu')(fu')^{-1}(gv')(hv')^{-1} = (eu')(hv')^{-1}$.

One question is whether u, v, u', v' may be chosen to be homogeneous elements of R . This is straightforward to check:

Write $u = \sum_{i=s}^{\infty} u_i$ and $v = \sum_{j=t}^{\infty} v_j$ (finite sums) with $u_s, v_t \neq 0$ (the pieces of u and v of lowest degree). Now, fu_s is the lowest degree piece of fu because f is homogeneous and R is a (graded) domain. Similarly, hv_t is the lowest degree piece of hv . Finally, $fu = hv \Rightarrow fu_s = hv_t$ again by the grading of R . Hence we may assume u and v (and u' and v') are homogeneous elements of R .

Next, we must ask whether the resulting fractions in (+) and (\times) above belong to D_0 . Again, this is easy to check, and we do so only for (+):

In the case of (+) we have $\deg e + \deg u = \deg f + \deg u = \deg h + \deg v = \deg g + \deg v$, so $\deg(eu + gv) = \deg(hv)$ as needed. \square

For what remains, we will need a stronger version of the q -commuting property than was proved above. Specifically, we need the following identity.

Proposition 30 *Put $f_{[-d]} := f_{\{n-d+1, \dots, n\}}$. Then for all $I \in \binom{[n]}{d}$, we have*

$$f_{[-d]}f_I = q^{|[-d] \setminus I|} f_I f_{[-d]}.$$

One can find a proof of this well-known identity in [16], which, after the specialization results of the previous section, we are now free to use.

Define $\mathcal{D}_0 \subset \mathcal{D}$ as in the proposition. This is the algebra of functions we seek. We must show that: (i) \mathcal{D}_0 is \sim -invariant; (ii) it is neither too big nor too

small inside \mathcal{D} .

Invariance:

Write \hat{f}_I for $f_I f_{[-d]}^{-1}$ inside \mathcal{D} . Note that \hat{f}_I is \sim -invariant. Finally, take $F = GH^{-1} \in \mathcal{D}_0$ (with $\deg G = \deg H = b$), and write $GH^{-1} = (Gf_{[-d]}^{-b})(Hf_{[-d]}^{-b})^{-1} = \widehat{G}\widehat{H}^{-1}$ in \mathcal{D} . Here we have written \widehat{G} for the rearrangement of $Gf_{[-d]}^{-b}$ putting one factor of $f_{[-d]}^{-1}$ to the right of each symbol f_I appearing in G . Then $GH^{-1}(Ag) = \widehat{G}\widehat{H}^{-1}(Ag) = \widehat{G}\widehat{H}^{-1}(A) = GH^{-1}(A)$ as needed.

Correct Size:

We look at the fields of fractions on the affine pieces of our projective space $Gr_q(d, n)$. \mathcal{D}_0 should contain them all, and be no bigger than necessary. Consider the “affine patch” of points $X_{[-d]} = \{ \{ |A_I|_q \} : |A_{\{n-d+1, \dots, n\}}|_q \neq 0 \}$ inside $Gr_q(d, n)$; $f_{[-d]}^{-1}$ is a well-defined function here. Moreover, by property (iii) of \sim we have $f_I f_{[-d]}^{-1}(Ag) = f_I f_{[-d]}^{-1}(A)$ when $Ag \sim A$. So we may consider the subalgebra \mathcal{A} of \mathcal{D} generated by $f_I f_{[-d]}^{-1}$ as a piece of the field of \sim -invariant functions we’re looking for. By the previous proposition, we may write every element of \mathcal{A} as $Gf_{[-d]}^{-\deg G}$ in \mathcal{D} , where G is a homogeneous polynomial in $\mathcal{G}_q(d, n)$. Finally, \mathcal{A} is noetherian (cf. [12], Theorem 1.4), so we may consider its right field of fractions $ff\mathcal{A} \subseteq \mathcal{D}$. Observe that $\mathcal{D}_0 \subseteq ff\mathcal{A}$: given $GH^{-1} \in \mathcal{D}_0$, we have

$$GH^{-1} = (Gf_{[-d]}^b) \cdot (Hf_{[-d]}^b)^{-1} \in ff\mathcal{A}.$$

On the other hand, note that all rings corresponding to all affine patches are subalgebras of \mathcal{D}_0 , and thus so are their fields of fractions—to whatever extent they exist. So we arrive at the natural

Definition 31 *The field of functions on $Gr_q(d, n)$ is the subfield \mathcal{D}_0 of \mathcal{D} generated by all elements $G \cdot H^{-1}$ with $G, H \in \mathcal{G}_q(d, n)$ homogeneous of the same degree.*

Proposition 32 *If $F \in \mathcal{D}_0$, then F is a rational function in $\{(f_I)(f_J)^{-1} : |I \cap J| = d - 1\}$.*

Remark. “ \sim ” is too strict a relation to allow Gaussian elimination... a procedure necessary in the proof of Proposition 26, so we cannot simply pass from quasi- to quantum- in that proposition.

PROOF, Sketch. The proof comes from the special form F takes. Let’s consider the commutative case for a moment. Start from $F = G/H$ with G and H homogeneous of the same degree, b say. Here one may divide the top and bottom by $f_{[-d]}^b$ and “interpolate” between the coordinate functions f_I occuring in G and H to get this same result in a more elementary fashion.

Example 33

$$\begin{aligned} \frac{f_{\{346\}} + f_{\{123\}}}{f_{\{135\}}} &= \frac{f_{\{346\}}f_{\{456\}}^{-1} + f_{\{123\}}f_{\{456\}}^{-1}}{f_{\{135\}}f_{\{456\}}^{-1}} \\ &= \frac{(f_{\{346\}}f_{\{456\}}^{-1}) + (f_{\{123\}}f_{\{126\}}^{-1})(f_{\{126\}}f_{\{156\}}^{-1})(f_{\{156\}}f_{\{456\}}^{-1})}{(f_{\{135\}}f_{\{345\}}^{-1})(f_{\{345\}}f_{\{456\}}^{-1})}. \end{aligned}$$

In the quantum setting, the same argument works as $f_{[-d]}$ q -commutes with every other coordinate function. \square

We have given some motivation for the further study of \mathcal{D}_0 . We conclude this section by showing that, like $Gr_q(d, n)$, it's behavior is governed by its quasi-counterpart.

Theorem 34 *If $F \in \mathcal{D}_0$ is identically zero on $Gr_q(d, n)$, then F is zero as a consequence of quasi-Plücker coordinate considerations.*

PROOF. Let $Y_{I,J}$ denote the right-hand side of (9) and $P_{i_1, I_{(1)}, J}$ denote the left-hand side of (12)—so $Y_{I,J} = 0$ in $\mathcal{G}_q(d, n)$, and $1 - P_{i_1, I_{(1)}, J} = 0$ in \mathcal{D}_0 . For $F \in \mathcal{D}_0$, write $F = GH^{-1}$ as above, with $G(\Gamma_q) = 0, H(\Gamma_q) \neq 0$. Then G —by the quantized basis theorem—is in the ideal generated by relations of type $(\mathcal{Y}_{I,J}^*)_{(1)}$. Write G as such, then consider $\bar{G} \in \mathcal{D}_0$ built from G by factoring each expression $w(Y_{I,J}^*)w'$ occurring as $wf_{\{i_1 \dots i_d\}}(1 - P_{i_1, I_{(1)}, J})f_{\{i_{d+1} \dots i_{d+r} j_1 \dots j_{d-r}\}}w'$ in the manner carried out in the proof of Theorem 28. \square

5 Future Steps

As mentioned earlier, we anticipate following this paper with another addressing more general quantum flags. Already from the results of this paper, one may confidently go on to create Grassmannians in other noncommutative settings where amenable determinants exist (e.g. superalgebras).

Beyond “specializations” such as those above, it would be interesting to study the ring of quasi-Plücker coordinates itself. Recall the classical result: *the homogeneous coordinate ring for the flag variety is a model for the irreducible polynomial representations of GL_n* . One challenge would be to use the quasi-Plücker coordinates to construct a noncommutative representation theory.

Acknowledgement

The author would like to thank Vladimir Retakh and Robert Wilson for many helpful discussions, and for encouraging the writing of this paper.

References

- [1] Paul Moritz Cohn, *Skew field constructions*, London Mathematical Society Lecture Note Series, no. 27, Cambridge University Press, Cambridge, 1977.
- [2] Pavel Etingof, Israel Gelfand, and Vladimir Retakh, *Factorization of differential operators, quasideterminants, and nonabelian Toda field equations*, Math. Res. Lett. **4** (1997), no. 2-3, 413–425.
- [3] R. Fioresi, *Quantum deformation of the Grassmannian manifold*, J. Algebra **214** (1999), no. 2, 418–447.
- [4] I. M. Gel'fand and V. S. Retakh, *Determinants of matrices over noncommutative rings*, Funktsional. Anal. i Prilozhen. **25** (1991), no. 2, 13–25, 96.
- [5] I. M. Gelfand and V. S. Retakh, *Quasideterminants, I*, Selecta Math. (N.S.) **3** (1997), no. 4, 517–546.
- [6] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson, *Quasideterminants*, Adv. in Math. **193** (2005), no. 1, 56–141.
- [7] Israel Gelfand and Vladimir Retakh, *Noncommutative Vieta theorem and symmetric functions*, The Gelfand Mathematical Seminars, 1993–1995, Birkhäuser Boston, Boston, MA, 1996, pp. 93–100.
- [8] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348.
- [9] K. R. Goodearl and T. H. Lenagan, *Quantum determinantal ideals*, Duke Math. J. **103** (2000), no. 1, 165–190.
- [10] W. V. D. Hodge, *Some enumerative results in the theory of forms*, Proc. Cambridge Philos. Soc. **39** (1943), 22–30.
- [11] W. V. D. Hodge and D. Pedoe, *Methods of algebraic geometry. Vol. I*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1994, Book I: Algebraic preliminaries, Book II: Projective space, Reprint of the 1947 original.
- [12] A. C. Kelly, T. H. Lenagan, and L. Rigal, *Ring theoretic properties of quantum Grassmannians*, J. Algebra Appl. **3** (2004), no. 1, 9–30.

- [13] Daniel Kroh and Bernard Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. **169** (1995), no. 1, 1–23.
- [14] V. Lakshmibai and N. Reshetikhin, *Quantum deformations of SL_n/B and its Schubert varieties*, Special functions (Okayama, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 149–168.
- [15] ———, *Quantum flag and Schubert schemes*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), Contemp. Math., vol. 134, Amer. Math. Soc., Providence, RI, 1992, pp. 145–181.
- [16] Bernard Leclerc and Andrei Zelevinsky, *Quasicommuting families of quantum Plücker coordinates*, Kirillov’s seminar on representation theory, Amer. Math. Soc. Transl. Ser. 2, vol. 181, Amer. Math. Soc., Providence, RI, 1998, pp. 85–108.
- [17] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.
- [18] Alexander Molev and Vladimir Retakh, *Quasideterminants and Casimir elements for the general linear Lie superalgebra*, Int. Math. Res. Not. (2004), no. 13, 611–619.
- [19] Christian Ohn, *“Classical” flag varieties for quantum groups: the standard quantum $SL(n, \mathbf{C})$* , Adv. Math. **171** (2002), no. 1, 103–138.
- [20] N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz **1** (1989), no. 1, 178–206, English transl.: *Leningrad Math. J.* **1** (1990), no. 1, 193–225.
- [21] Zoran Škoda, *Localizations for construction of quantum coset spaces*, Noncommutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., vol. 61, Polish Acad. Sci., Warsaw, 2003, pp. 265–298.
- [22] Earl Taft and Jacob Towber, *Quantum deformation of flag schemes and Grassmann schemes, I. A q -deformation of the shape-algebra for $GL(n)$* , J. Algebra **142** (1991), no. 1, 1–36.
- [23] Mitsuhiro Takeuchi, *A short course on quantum matrices*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, Notes taken by Bernd Strüber, pp. 383–435.
- [24] Jacob Towber, *Young symmetry, the flag manifold, and representations of $GL(n)$* , J. Algebra **61** (1979), no. 2, 414–462.